Propagation of Spatial Interaction under Superposition

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Abstract

We show that the superposition of two independent Markov point processes with respect to the same neighbourhood relation exhibits no second order interactions, and no third order ones if the point processes are identically distributed. In the limit, the independent superposition of standardised Markov point processes converges weakly to a Poisson process.

1 Introduction

(*Stoyan, Kendall and Mecke, 1995*) list the following three fundamental operations that can be performed on point patterns:

- thinning;
- clustering;
- superposition.

These operations allow the construction of new, more complex models from simpler ones, and as such are very useful in the modelling of spatial patterns.

A *thinning* operator returns a subset of the input pattern according to some deterministic or stochastic rule. As a simple example, each point may be retained independently with some fixed probability. More generally, the retention probability could depend on the location to take into account spatial inhomogeneity, or even on the rest of the pattern in case of dependent thinning.

A *cluster process* is a useful model for many natural phenomena of an aggregated or evolutionary nature. Here, the input pattern is interpreted as a collection of parent points, each giving rise to a cluster of daughter points centred around the parent. The output process is the ensemble of daughters. Note that although the terminology is biological, cluster processes arise in many contexts. For instance the well-known Neyman-Scott process was first proposed to model galaxies in space (*Neyman and Scott, 1958*). More precisely, under this model, the stars are scattered independently around a Poisson 'parent' process according to a given probability distribution.

Finally, the *superposition* operator takes two point processes and forms their union. For more information, see (*Stoyan, Kendall and Mecke, 1995*) or (*Daley and Vere-Jones, 1988*).

The simplest model for a random point configuration is a homogeneous Poisson process formalising complete spatial randomness. If such a process is independently thinned, the result is another Poisson process, possibly inhomogeneous if the retention probability depends on the location. The superposition of two independent Poisson processes is also a Poisson process, and, as we saw above, independent clustering with respect to a Poisson parent process yields a Neyman–Scott process.

In this paper, we shall take the class of Ripley–Kelly Markov point processes (*Ripley and Kelly, 1977*) as our building blocks. The latter are generalisations of the Poisson model allowing for local dependence between the points, and are widely used in practice (*Van Lieshout, 2000*). The effect of independent clustering on the Markov property was investigated by (*Baddeley, Van Lieshout and Møller, 1996*). Since an independent thinning can be seen as a cluster process in which each parent has at most a single daughter, their results are valid for the thinning operator as well. It was found that even a Neyman–Scott process with uniformly bounded clusters is not (in general) a Markov point process. However, if the parent process is Markov and the associated clusters are uniformly bounded and almost surely non-empty, then the resulting cluster process satisfies a weaker, connected component Markov property (*Baddeley and Møller, 1989*). For independent thinning, not even the connected component Markov property is preserved. For details, see (*Baddeley, Van Lieshout and Møller, 1996*).

Recently, (*Chin and Baddeley, 1999*) showed that the class of connected component Markov point processes is closed under independent superposition, hence a fortiori superposition of two Ripley–Kelly Markov point processes yields a connected component Markov point process. Here, we investigate in how far the Ripley–Kelly Markov property is preserved.

The plan of this paper is as follows. In section 2, some key results from the theory of Markov point processes are reviewed. In section 3, the interaction functions of the superposition of two independent Markov point processes are computed. The results are used to show that the Hammersley–Clifford factorisation is preserved up to second order, and that if the processes are identically distributed, the third order interaction structure is preserved as well. Section 4 considers the asymptotical behaviour, and section 5 is devoted to discussion and conclusions.

2 Set-up and notation

Throughout this paper, we will consider a finite point process X on a compact subset A of d-dimensional Euclidean space with non-trivial interior, so that the d-volume $\mu(A)$ is strictly positive and finite. The realisations of X are finite subsets $\mathbf{x} = \{x_1, \ldots, x_n\}$ $(n = 0, 1, \ldots)$ of A called *configurations*; the class of all configurations will be denoted by $\mathcal{C}$.

In order to define a probability distribution for X, we specify its density $p : \mathcal{C} \to [0, \infty)$. Thus, the probability that X contains exactly n points is

$$\frac{e^{-\mu(A)}}{n!} \int_A \cdots \int_A p(\{x_1, \ldots, x_n\})\, dx_1 \ldots dx_n$$

and given there are n points, the joint probability density of the locations is

$$p_n(x_1, \ldots, x_n) = \frac{p(\{x_1, \ldots, x_n\})}{\int_A \cdots \int_A p(\{y_1, \ldots, y_n\})\, dy_1 \ldots dy_n}.$$

To introduce interaction between the points, let $\sim$ be a reflexive, symmetric relation on A. For instance, in the *fixed range relation* two points $a, b \in A$ are related if and only if the distance between them does not exceed some $R > 0$. The point process X is *Markov with respect to* $\sim$ if its density $p(\cdot)$ is *hereditary*, that is $p(\mathbf{x}) > 0$ implies $p(\mathbf{y}) > 0$ for all $\mathbf{y} \subseteq \mathbf{x}$, and satisfies the Markov property that for any configuration $\mathbf{x}$ such that $p(\mathbf{x}) > 0$ and any $a \in A \setminus \mathbf{x}$ the likelihood ratio

$$\lambda(a \mid \mathbf{x}) = \frac{p(\mathbf{x} \cup \{a\})}{p(\mathbf{x})} \tag{1}$$

depends only on a and on $\{x_i \in \mathbf{x} : a \sim x_i\}$, the set of *neighbours* of a. The function $\lambda(\cdot \mid \cdot)$ is called the *Papangelou conditional intensity*; it can be interpreted as the conditional probability of finding a point at a given that the configuration elsewhere equals $\mathbf{x} \setminus \{a\}$.

The Hammersley–Clifford theorem (*Ripley and Kelly, 1977*) provides a convenient factorisation of $p(\cdot)$ into local interaction functions. Recall that a clique is a configuration $\mathbf{c}$ for which all its members are neighbours, i.e. $c \sim d$ for all $c, d \in \mathbf{c}$. By convention, the empty set and singletons are cliques. Now, $p(\cdot)$ defines a Markov point process if and only if it can be written as

$$p(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \phi(\mathbf{y}) \tag{2}$$

where $\phi(\mathbf{y}) = 1$ unless $\mathbf{y}$ is a clique. Thus, the Hammersley–Clifford factorisation in effect breaks up a high-dimensional joint probability density into manageable clique interaction functions $\phi(\cdot)$ that are easier to interpret and have a lower dimension.

The interested reader is referred to (*Ripley and Kelly, 1977*) or (*Van Lieshout, 2000*) for more details on Markov point processes.

3 Superposition

Let X_1 and X_2 be independent Markov point processes with respect to some neighbourhood relation $\sim$ on A, and write $p_i(\cdot)$ for the density of X_i. Then the superposition $X_s = X_1 \cup X_2$ has density

$$\begin{aligned} p_s(\mathbf{x}) &= e^{-\mu(A)} \sum_{\mathbf{x}_1, \mathbf{x}_2} p_1(\mathbf{x}_1)\, p_2(\mathbf{x}_2) \\ &= e^{-\mu(A)} \sum_{\mathbf{x}_1, \mathbf{x}_2} \left[\prod_{\mathbf{u} \subseteq \mathbf{x}_1} \phi_1(\mathbf{u}) \prod_{\mathbf{v} \subseteq \mathbf{x}_2} \phi_2(\mathbf{v}) \right] \end{aligned} \tag{3}$$

for $\mathbf{x} \in \mathcal{C}$. Here $\phi_i(\cdot)$ denote the interaction functions of X_i, $i \in \{1, 2\}$, and the sum ranges over all ordered partitions of $\mathbf{x}$ in two components $\mathbf{x}_1$ and $\mathbf{x}_2$. By theorem 3 in (*Chin and Baddeley, 1999*), the superposition density $p_s(\cdot)$ factorises into a product over terms associated with each $\sim$-connected component (cf. (*Baddeley and Møller, 1989*)). Here we will show that in general $p_s(\cdot)$ fails to satisfy the Hammersley–Clifford factorisation of Eq.(2), but that the pair-interaction function reduces to 1 for non-cliques, as does the third order interaction function if X_1 and X_2 are identically distributed.

Counterexample 1 Let $\sim$ be a reflexive, symmetric neighbourhood relation on A. Suppose X_1 and X_2 are independent, identically distributed Strauss processes (*Strauss, 1975*) with density

$$p(\mathbf{x}) = \alpha \gamma^{s(\mathbf{x})}$$

where $\gamma \in (0,1)$ is a repulsion parameter, and $s(\mathbf{x})$ denotes the number of neighbour pairs in $\mathbf{x}$. Assume A is sufficiently large to allow for a configuration $\mathbf{x} = \{a, b, c, d\}$ such that $a \sim b \sim c \sim d$ are the only related points (see figure 1). Then the Papangelou conditional intensity $\lambda_s(\cdot \mid \cdot)$ of the superposition $X_s = X_1 \cup X_2$ satisfies

$$\lambda_s(d \mid \{a,b,c\}) = \frac{2\gamma^4 + 12\gamma^2 + 2}{2\gamma^2 + 4\gamma + 2} \neq \frac{2\gamma^2 + 4\gamma + 2}{4} = \lambda_s(d \mid \{a,c\}),$$

hence X_s is not Markovian with respect to the given relation.

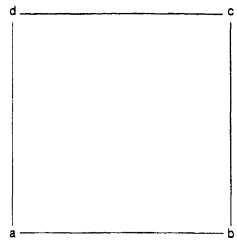

Figure 1: Neighbourhood graph on $\mathbf{x} = \{a, b, c, d\}$.

Surprisingly, one has to consider sets of four points in the above counterexample; the pair and triple 'interactions' do respect the neighbourhood relation. To make this statement more precise, define 'interaction functions' recursively as follows.

Definition 1 *Let X_1 and X_2 be independent Markov point processes with respect to a reflexive, symmetric relation $\sim$ on A, and let X_s be the superposition of X_1 and X_2. Recursively define*

$$\phi_s(\emptyset) = p_s(\emptyset)$$

and

$$\phi_s(\mathbf{x}) = \frac{p_s(\mathbf{x})}{\prod_{\mathbf{x} \neq \mathbf{y} \subset \mathbf{x}} \phi_s(\mathbf{y})} \tag{4}$$

for non-empty configurations $\mathbf{x} \in \mathcal{C}$ (setting $0/0 = 0$ if $\mathbf{x}$ is a $\sim$-clique and 1 otherwise).

Lemma 1 *Let X_1 and X_2 be independent Markov point processes with respect to a reflexive, symmetric relation $\sim$ on A. Then the density $p_s(\cdot)$ of the superposition $X_s = X_1 \cup X_2$ is hereditary, and $\phi_s(\cdot)$ is well-defined.*

Proof: Suppose $p_s(\mathbf{x})$ is strictly positive for some configuration $\mathbf{x}$. By Eq.(3), a partition $\mathbf{x}_1 \cup \mathbf{x}_2 = \mathbf{x}$ exists for which $p_1(\mathbf{x}_1) > 0$ and $p_2(\mathbf{x}_2) > 0$. Since X_1 and X_2 are Markov point processes, $p_i(\mathbf{x}_i \cap \mathbf{y})$, $i \in \{1,2\}$, is strictly positive for every configuration $\mathbf{y} \subseteq \mathbf{x}$. Therefore

$$p_s(\mathbf{y}) \geq e^{-\mu(A)}\, p_1(\mathbf{x}_1 \cap \mathbf{y})\, p_2(\mathbf{x}_2 \cap \mathbf{y}) > 0$$

for all $\mathbf{y} \subseteq \mathbf{x}$, hence $p_s(\cdot)$ is hereditary.

To show that $\phi_s(\cdot)$ is well-defined, suppose $\prod_{\mathbf{x} \neq \mathbf{y} \subset \mathbf{x}} \phi_s(\mathbf{y}) = 0$. Without loss of generality, let $\mathbf{y}$ be a proper subset of $\mathbf{x}$ such that $\phi_s(\mathbf{y}) = 0$ and that is smallest in the sense that no subconfiguration $\mathbf{y} \neq \mathbf{z} \subset \mathbf{y}$ has a vanishing interaction function $\phi_s(\mathbf{z}) = 0$. By Eq.(4), $p_s(\mathbf{y}) = 0$. Finally, as $p_s(\cdot)$ is hereditary, $p_s(\mathbf{x}) = 0$, and $\phi_s(\mathbf{x})$ is well-defined. □

We are now ready to state the main theorem.

Theorem 1 *Let X_1 and X_2 be independent and identically distributed Markov point processes with respect to a reflexive, symmetric relation $\sim$ on A, specified by their density $p(\cdot)$. Then, for configurations $\mathbf{x} \in \mathcal{C}$ containing at most three points, the superposition density $p_s(\mathbf{x})$ at $\mathbf{x}$ satisfies the Hammersley–Clifford factorisation, that is*

$$p_s(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \phi_s(\mathbf{y}),$$

and $\phi_s(\mathbf{x}) = 1$ whenever $\mathbf{x}$ is not a $\sim$-clique.

Proof: Write $p(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \phi(\mathbf{y})$ for the factorisation of $p(\cdot)$ over cliques. Substitution of Eq.(3) into Eq.(4) and the fact that $\phi_s(\emptyset) > 0$ yield

$$\phi_s(\emptyset) = e^{-\mu(A)}\, \phi(\emptyset)^2; \tag{5}$$

$$\phi_s(\{\xi\}) = 2\phi(\{\xi\}). \tag{6}$$

Furthermore, for $\xi \neq \eta \in A$,

$$p_s(\{\xi, \eta\}) = \frac{1}{2}\, \phi_s(\emptyset)\, \phi_s(\{\xi\})\, \phi_s(\{\eta\})\, [1 + \phi(\{\xi, \eta\})].$$

Clearly, either $\phi_s(\{\xi\}) = 0$ or $\phi_s(\{\eta\}) = 0$ implies $p_s(\{\xi, \eta\}) = 0$, hence by Eqs.(4)–(6)

$$\phi_s(\{\xi, \eta\}) = \begin{cases} 1\,\{\xi \not\sim \eta\} & \text{if } \phi(\xi) = 0 \text{ or } \phi(\eta) = 0 \\ 1 + \frac{1}{2}(\phi(\{\xi, \eta\}) - 1) & \text{else} \end{cases} \tag{7}$$

from which the required factorisation follows for doublets.

In order to compute the third order interaction function $\phi_s(\cdot)$, note that for distinct $\xi, \eta, \zeta \in A$,

$$p_s(\{\xi, \eta, \zeta\}) = 2\phi_s(\emptyset)\, \phi(\{\xi\})\, \phi(\{\eta\})\, \phi(\{\zeta\})[$$
$$\phi(\{\xi, \eta\})\, \phi(\{\xi, \zeta\})\, \phi(\{\eta, \zeta\})\, \phi(\{\xi, \eta, \zeta\}) + \phi(\{\xi, \eta\} + \phi(\{\xi, \zeta\}) + \phi(\{\eta, \zeta\})].$$

Assuming $\prod_{\mathbf{y} \subset \{\xi, \eta, \zeta\}} \phi_s(\mathbf{y}) > 0$, by Eqs.(4)–(7),

$$\begin{aligned} \phi_s(\{\xi, \eta, \zeta\}) &= 1 + \frac{1}{4} \frac{\phi(\{\xi, \eta\})\, \phi(\{\xi, \zeta\})\, \phi(\{\eta, \zeta\})}{\phi_s(\{\xi, \eta\})\, \phi_s(\{\xi, \zeta\})\, \phi_s(\{\eta, \zeta\})} (\phi(\{\xi, \eta, \zeta\}) - 1) \\ &+ \frac{1}{8} \frac{(\phi(\{\xi, \eta\}) - 1)\, (\phi(\{\xi, \zeta\}) - 1)\, (\phi(\{\eta, \zeta\}) - 1)}{\phi_s(\{\xi, \eta\})\, \phi_s(\{\xi, \zeta\})\, \phi_s(\{\eta, \zeta\})}. \end{aligned} \tag{8}$$

If $\{\xi, \eta, \zeta\}$ is not a clique, $\phi(\{\xi, \eta, \zeta\}) = 1$. Furthermore, at least one of $\phi(\{\xi, \eta\})$, $\phi(\{\xi, \zeta\})$, $\phi(\{\eta, \zeta\})$ takes the value 1. Hence Eq.(8) reduces to 1 as well. It remains to

consider the case $\prod_{\mathbf{y}\subset\{\xi,\eta,\zeta\}} \phi_s(\mathbf{y}) = 0$. If a first order term equals zero, $p_s(\{\xi,\eta,\zeta\}) = 0$ so $\phi_s(\{\xi,\eta,\zeta\})$ is the clique indicator function. If the first order terms are all strictly positive, a second order term, say $\phi_s(\{\xi,\eta\})$, must be zero. But then $\phi(\{\xi,\eta\}) = -1$, contradicting the fact that interaction functions are non-negative. In summary, the Hammersley–Clifford factorisation holds for all configurations consisting of at most three points, which completes the proof. □

If the components are not identically distributed, the Hammersley–Clifford factorisation does not hold for triples. A counterexample is obtained by considering the superposition of two independent Strauss processes with *different* repulsion parameters. The example should be compared to counterexample 1.

Counterexample 2 Let $\sim$ be a reflexive, symmetric neighbourhood relation on A. Consider two independent Strauss processes (*Strauss, 1975*) X_1 and X_2 defined by their densities

$$p_i(\mathbf{x}) = \alpha_i \gamma_i^{s(\mathbf{x}_i)} \qquad i = 1, 2$$

with repulsion parameters $\gamma_1 \neq \gamma_2 \in (0,1)$. The exponent $s(\mathbf{x}_i)$ denotes the number of neighbour pairs in $\mathbf{x}_i$ $(i = 1, 2)$. Suppose A is sufficiently large to allow for a configuration $\{\xi,\eta,\zeta\}$ for which $\xi \sim \eta \sim \zeta$ but $\xi \not\sim \zeta$. Then

$$p_s(\{\xi,\eta,\zeta\}) = \phi_s(\emptyset)\left((\gamma_1+1)^2 + (\gamma_2+1)^2\right),$$

which implies

$$\phi_s(\{\xi,\eta,\zeta\}) = 2\frac{(\gamma_1+1)^2 + (\gamma_2+1)^2}{(\gamma_1+\gamma_2+2)^2}.$$

Hence $\phi_s(\{\xi,\eta,\zeta\}) \neq 1$, unless $\gamma_1 = \gamma_2$.

4 Asymptotics

In this section, let us consider the superposition of a large number of independent realisations X_i, $i = 1, 2, \ldots$, of a Markov point process X with density $p(\mathbf{x}) = \prod_{\mathbf{y}\subseteq\mathbf{x}} \phi(\mathbf{x})$. Write $\phi_n(\cdot)$ for the interaction function of $\bigcup_{i=1}^n X_i$, and assume that the first order interaction function of X is strictly positive. Then, by iterating Eqs.(5)–(7), one obtains

$$\phi_n(\{\xi\}) = n\phi(\{\xi\}); \tag{9}$$

$$\phi_n(\{\xi,\eta\}) = 1 + \frac{1}{n}(\phi(\{\xi,\eta\}) - 1). \tag{10}$$

Therefore, as n tends to infinity, $\phi_n(\{\xi,\eta\})$ tends to 1. Intuitively this means that each time a new X_{n+1} is added, the intensity increases while the inter-point interactions grow weaker. Thus, if the intensity is rescaled by a factor $1/n$ to stabilise the mean number of points, one would expect to loose all inter-point dependence in the limit. The remainder of the section is devoted to making this claim more precise.

Consider a sequence $\tilde{X}^{(n)}$ of point process obtained as the superposition of n independent, identically distributed Markov point processes $\tilde{X}_{n1}, \ldots, \tilde{X}_{nn}$ with density

$$\tilde{p}_n(\mathbf{x}) = \frac{\tilde{\alpha}_n}{\phi(\emptyset)}\left(\frac{1}{n}\right)^{n(\mathbf{x})} p(\mathbf{x}).$$

As before, $n(\mathbf{x})$ denotes the number of points in the configuration $\mathbf{x}$, and $\tilde{\alpha}_n$ is a normalisation constant. Hence $\tilde{X}_{ni}$ has the same second and higher order interaction functions as X, but the first order interaction terms are standardised to avoid explosion (cf. Eq.(9)). Write $N_{\tilde{X}_{n1}}(B)$ for the random variable that counts the number of points of $\tilde{X}_{n1}$ in the bounded Borel set $B \subseteq A$. Then the following properties hold.

Lemma 2 *The triangular array of point processes $\tilde{X}_{ni}$ is asymptotically negligable, that is*

$$\lim_{n\to\infty} P\left(N_{\tilde{X}_{n1}}(B) > 0\right) = 0$$

for all bounded Borel sets $B \subseteq A$. Furthermore, for every ϕ-continuity set $B \subseteq A$,

$$\lim_{n\to\infty} n\,P\left(N_{\tilde{X}_{n1}}(B) \geq 2\right) = 0; \qquad \lim_{n\to\infty} n\,P\left(N_{\tilde{X}_{n1}}(B) \geq 1\right) = \int_B \phi(\{\xi\})\,d\xi.$$

Proof: Write $E_{p(\cdot)}$ for the expectation with respect to $p(\cdot)$, and $n(X)$ for the cardinality of the random configuration X. Note that

$$\tilde{\alpha}_n^{-1} = E_{p(\cdot)}\left[\frac{1}{\phi(\emptyset)}\left(\frac{1}{n}\right)^{n(X)}\right] \to e^{-\mu(A)} \qquad (n \to \infty)$$

by the dominated convergence theorem. Hence

$$P(N_{\tilde{X}_{n1}}(B) > 0) \leq P(N_{X_{n1}}(A) > 0) = 1 - \tilde{\alpha}_n\, e^{-\mu(A)} \to 0$$

as n tends to infinity. Similarly,

$$n\,P(N_{\tilde{X}_{n1}}(B) \geq 2) = E_{p(\cdot)}\left[\frac{\tilde{\alpha}_n}{\phi(\emptyset)}\left(\frac{1}{n}\right)^{n(X)-1} 1\{n(X \cap B) \geq 2\}\right] \to 0$$

and

$$\begin{aligned} n\,P(N_{\tilde{X}_{n1}}(B) = 1) &= E_{p(\cdot)}\left[\frac{\tilde{\alpha}_n}{\phi(\emptyset)}\left(\frac{1}{n}\right)^{n(X)-1} 1\{n(X \cap B) = 1\}\right] \\ &\to E_{p(\cdot)}\left[\frac{e^{\mu(A)}}{\phi(\emptyset)}\, 1\{n(X \cap B) = 1; n(X \cap A \setminus B) = 0\}\right] = \int_B \phi(\{\xi\})\,d\xi \end{aligned}$$

as $n \to \infty$. □

Combining lemma 2 with the Ososkov–Franken–Grigelionis theorem 9.2.V in (*Daley and Vere-Jones, 1988*) yields the following limit theorem.

Theorem 2 *Let X be a Markov point process with respect to a reflexive, symmetric relation $\sim$ on A, specified by its density $p(\mathbf{x})$. Let $\tilde{X}_{ni}$, $i = 1, \ldots, n$, be independent, identically distributed Markov point processes with density $\tilde{p}_n(\mathbf{x}) \propto \left(\frac{1}{n}\right)^{n(\mathbf{x})} p(\mathbf{x})$. Then, as $n \to \infty$, the superposition $\tilde{X}^{(n)} = \bigcup_{i=1}^n \tilde{X}_{ni}$ converges weakly to a Poisson point process on A with intensity function $\phi(\{\xi\})$, $\xi \in A$.*

5 Conclusion

The independent superposition of Markov point processes in general is not a Markov point process with respect to the same neighbourhood relation $\sim$ (as shown in counterexample 1). Indeed, higher order correlation is introduced. To see this, consider independent, identically distributed *pairwise interaction processes* X_1 and X_2 with density $p(\mathbf{x}) = \alpha\beta^{n(\mathbf{x})}\prod_{a,b\in\mathbf{x}}\gamma(a,b)$ where $\beta > 0$ is an intensity parameter, $\gamma : A \times A \to [0,1]$ a Borel measurable interaction function with $\gamma(a,b) = 1$ whenever $a \not\sim b$. Writing $\gamma_s(\xi,\eta) = 1 + \frac{1}{2}(\gamma(\xi,\eta) - 1)$, by Eq.(8) the superposition interaction function for a triple $\{x_1, x_2, x_3\}$ is given by

$$\phi_s(\{x_1,x_2,x_3\}) = 1 + \frac{1}{8}\frac{(\gamma(x_1,x_2)-1)(\gamma(x_1,x_3)-1)(\gamma(x_2,x_3)-1)}{(\gamma_s(x_1,x_2)-1)(\gamma_s(x_1,x_3)-1)(\gamma_s(x_2,x_3)-1)}$$

which is not necessarily identically 1.

On the other hand, lower order inter-point interactions are not introduced. More specifically, suppose that a Markov density is of the form

$$p(\mathbf{x}) = \alpha\prod_{x\in\mathbf{x}}\phi(\{x\})\prod_{\mathbf{y}\subseteq\mathbf{x};n(\mathbf{y})>k\geq 2}\phi(\mathbf{y})$$

then $\phi_s \equiv 1$ on $\mathcal{C}\cap\{\mathbf{y}\in\mathcal{C} : 2 \leq n(\mathbf{y}) \leq k\}$.

Finally, asymptotic results were discussed, showing that in the limit all interpoint interactions disappear.

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